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Transient conjugate condensation process on a vertical plate with finite thermal inertia

C. TREVIÑO and F. MÉNDEZ
Facultad de Ciencias, UNAM, 04510 México D.F., Mexico

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Abstract—In this paper we analyse the transient condensation process of saturated vapor in contact with one surface of a vertical thin plate, caused by a uniform cooling rate on the other surface of the plate, applied at a time $t = 0$. The effects of longitudinal and transversal heat conduction, as well as the thermal inertia for the plate, are considered. The nondimensional governing equations are reduced to a system of four partial differential equations with six nondimensional parameters. The transient evolution of the condensed layer thickness and the temperature of the plate are obtained using different realistic limits, including the cases of very good and poor conducting plates. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Since the classical work of Nusselt [1], the theoretical studies of laminar film condensation have received considerable attention in the literature. By equating the gravity and viscous forces, Nusselt obtained the condensed layer thickness of a saturated vapor in contact with a vertical plate of uniform temperature, showing good qualitative agreement with experimental observations for normal engineering fluids. In general, the development in this area has been concentrated to those investigations in which the relative importance of the additional complicating factors is revealed. In this sense, the natural convection condensation process on vertical plates is not an exceptional case and relevant analysis including inertia, convection and shear stress effects at the condensate surface, shows that the simple Nusselt results are surprisingly accurate over a wide range of conditions. Particular contribution on this issue comes from Rohsenow [2], where he modified Nusselt's analysis including the energy convection in the heat balance equation. However, his analysis did not include the inertial forces in a similar way, as was made by Bromley [3], by other alternative procedures.

In an effort to obtain a better approximation, Sparrow and Gregg [4], introducing a boundary layer treatment and similarity transformations of the governing equations, showed numerically that the inertial effects on heat transfer are not important if the Prandtl number is larger or equal to 10, and was quite small for even a Prandtl number of order unity. Later, Chen [5] solved integral forms of the boundary layer equations by perturbations methods, including the retarding effect of vapor shear stress on the condensate film. A comparison of the results obtained by Sparrow *et al.* [4] with those obtained by Chen [5] shows that the influence of surface shear stress is negligible at higher Prandtl numbers. In order to have a more accurate influence of this effect, Koh *et al.* [6] incorporated the interfacial shear stress through the use of simultaneous solution of the vapor and condensate boundary layer equations and concluded that the effect of the shear stress is only significant when the condensation rate is sufficiently high. Similar results were obtained by Rose [8], using a similarity approach confirming the previous problem solved by Chen and gave even more accurate expressions for the Nusselt number. The state-of-the-art of the laminar film condensation on vertical plates and other condensing processes can be found in ref. [7], and more recently in refs [8] and [9]. The foregoing studies are particularly applied to isothermal vertical plates, with known temperature. However, theoretical studies of film condensation processes with non-isothermal conditions have received little attention in the literature. In particular, Patankar and Sparrow [10] solved the problem of condensation on an extended surface by considering the heat conduction in a fin coupled with the condensation process. Their numerical solution of the governing equations confirms the physical influence of the non-isothermal extended surface over the condensing process. Subsequently, it was shown by Wilkins [11] that an explicit analytical solution is possible for the formulation of Patankar and Sparrow [10]. A main conclusion of this article is that the studies of condensation on extended surfaces, using the classical Nusselt analysis for an isothermal case, form a class by themselves and an estimation of the surface area requirements of the condenser is not appropriate. In order to extend these particular cases with non-isothermal conditions, Brouwers [12] performed recently an analysis of the condensation of a pure saturated vapor on a cooled channel plate, including the interaction between the cooling liquid, the condensate and the vapor. His numerical results confirm that this

NOMENCLATURE

c	specific heat
f	nondimensional stream function, defined in equation (21)
g	acceleration of gravity
h	thickness of the plate
h_{fg}	latent heat of condensation
Ja	Jakob number defined in equation (5)
L	length of the plate
Pr	Prandtl number, $Pr = \mu_l c_l/\lambda_l$
p, q	similarity variables defined in equation (53)
q_e	prescribed heat flux per unit length
q_l	actual heat flux per unit length at the liquid–vapor interface
S	Strouhal number defined in equation (17)
t	time
t_{90}	evolution time in physical units
t_d	characteristic time in the wall transversal direction
t_{dL}	characteristic time in the wall longitudinal direction
t_e	characteristic evolution time for the plate
t_l	characteristic residence time in the condensed fluid
T	temperature
T_r	characteristic temperature, $T_r = \alpha q_e h/\varepsilon^2\lambda_w$
T_s	temperature of the saturated vapor
u_c	characteristic longitudinal velocity of the condensed fluid
x, y	Cartesian coordinates
z	nondimensional transversal coordinate for the plate, $z = y/h$.

Greek symbols

α	heat conduction parameter defined in equation (12)
Γ	nondimensional mass flow rate defined in equation (28)
γ	nondimensional parameter defined in equation (4)
Δ	nondimensional thickness of the condensed layer, $\Delta = \delta(\chi, \tau)/\delta_{f\infty}$
Δ_f	nondimensional thickness of the condensed layer at the lower edge
δ	thickness of the condensed layer
$\delta_{f\infty}$	thickness of the steady-state condensed layer at $\chi = 1$, $\delta_{f\infty} = L(Ja/\gamma)^{1/4}$
η	nondimensional transversal coordinate, $\eta = y/\delta_{f\infty}$
ε	aspect ratio of the plate, $\varepsilon = h/L$
λ	thermal conductivity
μ_l	dynamic viscosity of the condensed fluid
ν_l	kinematic coefficient of viscosity of the condensed fluid
ρ	density
χ	nondimensional longitudinal coordinate, $\chi = x/L$
τ	nondimensional time, $\tau = \lambda_w t/(L^2\rho_w c_w \alpha)$
τ_{90}	nondimensional evolution time
θ	nondimensional temperature of the condensed fluid, $\theta = (T_s - T)/T_r$
θ_w	nondimensional temperature of the plate, $\theta_w = (T_s - T_w)/T_r$.

Subscripts

0	initial conditions
∞	steady-state conditions
f	conditions at the lower edge of the plate
l	conditions at the condensed fluid
w	conditions at the wall
s	conditions at the saturated vapor.

interaction has to be taken into account in order to have more realistic models in this type of process.

In relation with transient condensation process Sparrow and Siegel [13] studied the vertical laminar-film condensation for a sudden drop in wall temperature. They used the method of characteristics to obtain a closed form solution. Chung [14] used a perturbation analysis to extend Sparrow and Siegel's work [13] including nonlinear temperature and non-parabolic velocity profiles. In both works a change in temperature at the surface facing the saturated vapor was assumed. More recently Reed *et al.* [15] studied numerically the same problem, but included the complete energy and momentum equations for the condensed fluid, as well as interfacial shear stresses. In this work the transient terms arise from changing conditions in the vapor. In all of these works, the thermal inertia of the wall was not considered.

The main objective of this paper is to analyse, using asymptotic as well as numerical methods, the transient laminar film condensation process on a nonisothermal vertical flat plate with finite thermal conductivity and capacity. In most realistic cases, the thermal inertia of the wall where the condensation process is taking place, represents the limiting factor of the transient process. The wall energy governing equation is coupled with the condensing process at one vertical face of the flat plate, whereas the flat plate is cooled with a known constant external flux q_e at the other vertical face, which is applied at time $t = 0$.

2. ORDER OF MAGNITUDE ESTIMATES

The physical model under study is shown in Fig. 1. A thin vertical plate with an initial temperature T_{w0}, length L and thickness h, is placed to its right in a stagnant atmosphere filled with saturated vapor with a temperature T_s. Its upper right corner coincides with the origin of a Cartesian coordinate system whose y axis points in the direction normal to the plate and its x axis points down in the plate's longitudinal direction, that is in the direction of gravity. At time $t = 0$, a known heat flux per unit length, q_e, is applied from the other surface of the plate. A thin transient condensed layer develops with increasing thickness downstream falling by gravity. The density of the condensed fluid, ρ_l, is assumed to be constant and much larger than the vapor density. An order of magnitude analysis gives that the condensed fluid longitudinal velocity is of the order,

$$u_c \sim \frac{g}{\nu_l}\delta^2(x,t), \tag{1}$$

where $\delta(x, t)$ is the condensed layer thickness, g is the acceleration of gravity and ν_l corresponds to the kinematic coefficient of viscosity ($=\mu_l/\rho_l$). An overall condensed mass balance gives then the following relationship [13]:

$$\frac{q_l L}{h_{fg}} \sim \frac{\rho_l \delta L}{t_l} + \rho_l u_c \delta. \tag{2}$$

The first term at the right hand side corresponds to the accumulation term, while the second term corresponds to the mass outflow. Here t_l is the characteristic time of the condensed film formation, h_{fg} denotes the latent heat of condensation and q_l is the actual heat flux per unit surface at the condensed liquid–vapor interface, $q_l = \lambda_l \partial T/\partial y|_s$, where λ_l corresponds to the thermal conductivity of the condensed fluid. From the final steady-state ($q_l \to q_e$ for $t \to \infty$) relationships (1) and (2), we obtain that the thickness of the condensed layer related to the length of the plate is given by

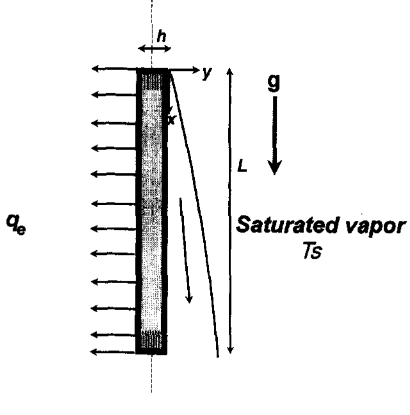

Fig. 1. Schematic diagram of the studied physical model.

$$\frac{\delta_{f\infty}}{L} \sim \left(\frac{Ja}{\gamma}\right)^{1/4}, \tag{3}$$

where γ is the still unnamed nondimensional parameter [16]

$$\gamma = \frac{gL^3}{\nu_l^2} \tag{4}$$

and Ja corresponds to the appropriate Jakob number and represents the ratio of the sensible heat energy absorbed by the liquid to the latent heat of the liquid during condensation, defined by

$$Ja = \left(\frac{3q_e L^{1/4}}{\rho_l h_{fg} g^{1/4} \nu^{1/2}}\right)^{4/3}. \tag{5}$$

A boundary layer for the condensed film is obtained in the limit $Ja/\gamma \to 0$, which is the case analysed in this work. The non-dimensional velocity or Reynolds number, $Re = u_c L/\nu_l$, associated to the condensation process, is of the order of $Re = O(\gamma Ja)^{1/2}$. The condensed fluid velocity is then of order

$$u_c \sim \sqrt{gLJa}. \tag{6}$$

Using the order of magnitude estimate (3), from relation (2) we obtain the order of magnitude of the characteristic time for the condensed film evolution, t_l, as

$$t_l \sim \frac{\nu_l L}{g\delta^2} \sim \sqrt{\frac{L}{gJa}}, \tag{7}$$

which is exactly the same as the residence time in the condensed film L/u_c. We will assume for simplicity that both edges of the plate are adiabatic. However, other boundary conditions could be assumed without difficulty. Therefore, the global heat flux difference between both surfaces is related to the accumulation term as

$$q_e - q_l \sim \frac{\rho_w c_w h(T_{w0} - T_{w\infty})}{t_e}, \tag{8}$$

and also

$$q_l + q_e \sim \frac{2\lambda_w \Delta T_w}{h}. \tag{9}$$

In these relationships, ρ_w, c_w and λ_w represent the density, specific heat and thermal conductivity of the wall material. ΔT_w is the characteristic transversal temperature drop at the wall; $T_{w\infty}$ corresponds to an average value of the wall temperature for $t \to \infty$, and t_e is the evolution time of the transient process. For large values of t, the total temperature change is then $\Delta T_\infty \sim \Delta T_{l\infty} + \Delta T_{w\infty}$ where $\Delta T_{l\infty}$ is the characteristic temperature drop in the transversal direction for the condensed fluid. The steady state version of the relationships (8) and (9) is

$$q_e = q_l = \frac{\lambda_l \Delta T_{l\infty}}{\delta_{f\infty}} = \frac{\lambda_w \Delta T_{w\infty}}{h}. \tag{10}$$

Therefore, the relative temperature drop transversally on the plate is given by

$$\frac{\Delta T_{w\infty}}{\Delta T_\infty} \sim \frac{1}{1+\alpha/\varepsilon^2}. \tag{11}$$

Here, ε is the aspect ratio of the plate given by $\varepsilon = h/L \ll 1$ and α is the heat conduction parameter which corresponds to the ratio of the heat conducted longitudinally by the plate to the heat convected from the condensed fluid and is defined by

$$\alpha = \frac{\lambda_w h}{\lambda_l L}\left(\frac{Ja}{\gamma}\right)^{1/4}. \tag{12}$$

For values of $\alpha/\varepsilon^2 \gg 1$, the transversal temperature variations in the plate are very small, of order ε^2/α at most. This represents the thermally thin wall limit, where the transversal temperature variations are negligible compared with the overall temperature drop, ΔT_∞. On the contrary, for values of α/ε^2 of order unity, most of the transversal temperature drop occurs in the solid. This is the thermally thick wall limit. The global temperature change is then

$$\Delta T_\infty \sim \frac{q_e h}{\lambda_w}\left(1+\frac{\alpha}{\varepsilon^2}\right). \tag{13}$$

Thus, the final value of the temperature at the middle plane of the plate can be written as

$$T_{w\infty} \sim T_s - \Delta T_\infty - \frac{\Delta T_{w\infty}}{2} \sim T_s - \frac{q_e h}{\lambda_w}\left(\frac{3}{2}+\frac{\alpha}{\varepsilon^2}\right). \tag{14}$$

From the order of magnitude estimates (14) and (8), we obtain the order of magnitude for t_e as

$$t_e \sim h\rho_w c_w\left[\frac{T_{w0}-T_s}{q_e} + \frac{h}{\lambda_w}\left(\frac{3}{2}+\frac{\alpha}{\varepsilon^2}\right)\right]. \tag{15}$$

We will assume, without any loss in generality, that the initial temperature of the plate corresponds to the temperature of the saturated vapor, $T_{w0} = T_s$. Therefore

$$t_e \sim t_d\left(\frac{3}{2}+\frac{\alpha}{\varepsilon^2}\right), \tag{16}$$

where t_d is the wall diffusion time in the transversal direction, $t_d \sim h^2\rho_u c_w/\lambda_w$. The corresponding wall diffusion time in the longitudinal direction is $t_{dL} \sim L^2\rho_w c_w/\lambda_w$. For large values of t_e/t_d, the temperature of the wall in the transversal direction becomes uniform very rapidly and then the thermally thin wall approximation can be used. In this case, the temperature of the wall depends mainly on the longitudinal coordinate and time and is justified for $\alpha/\varepsilon^2 \gg 1$. For large values of t_e/t_{dL}, the temperature of the wall becomes uniform spatially very rapid, thus depending exclusively on time. This ratio is given by $t_e/t_{dL} \sim \alpha$. For values of α of order unity or larger, the longitudinal heat conduction is very important and has to be retained in the analysis. However, for very small values of α, compared with unity, the longitudinal heat conduction is unable to compete with the heat convection and ceases to be important.

The ratio of the characteristic residence time in the condensed fluid t_l to the solid evolution time, t_e, represents an important parameter denoted by the Strouhal number, S defined by

$$S = \frac{t_l}{t_e} = \sqrt{\frac{L}{gJa}}\frac{\lambda_w}{L^2\rho_w c_w \alpha}. \tag{17}$$

For $S \ll 1$, the condensed film can be treated as quasi-steady and the limiting factor corresponds to the thermal inertia of the wall. On the other hand, for $S \gg 1$, the wall can be assumed to be quasi-steady and the process is controlled by the condensed film production.

In this paper, we will study the case $Ja \ll 1$, which in many applications is fully justified, meaning that the sensible heat is much lower than the latent heat of condensation. Therefore, the production of condensate is rather small, generating a very thin liquid film. Typical values for γ are of order $\gamma \sim 10^{10}$. The condensed layer thickness is then typically $\delta \sim 10^{-3}L$, thus the boundary layer approximation can be applied. The ratio of the thermal conductivities λ_w/λ_l can reach values of the order of 10^3. Therefore, a typical value of α is $\alpha \sim h/L$ and $\alpha/\varepsilon^2 \sim L/h$. In Section 3 we deduce the governing equations. The thermally thin wall approximation is then applied and the asymptotic limits $\alpha \gg 1$ and $\alpha \ll 1$ are studied. The asymptotic limit $\alpha \gg 1$ is important for this problem, because we can obtain a closed form solution for the condensed layer thickness evolution, which gives accurate results for values of α of order unity. Regarding the values of S, typical values of S are of order $S \sim 10^{-3}/\alpha$, which is clearly much lower than unity for typical values of α. This means that the limiting or controlling factor for the transient processes is in most cases the wall heat capacity, not considered in previous mentioned works [13–15].

3. FORMULATION

The nondimensional governing equations for the plate and the condensed fluid are given by

solid

$$\frac{\partial^2\theta_w}{\partial\chi^2} + \frac{1}{\varepsilon^2}\frac{\partial^2\theta_w}{\partial z^2} = \frac{1}{\alpha}\frac{\partial\theta_w}{\partial\tau}. \tag{18}$$

condensed fluid

$$\frac{\partial^3 f}{\partial\eta^3} + 1 = Ja\left\{S\frac{\partial^2 f}{\partial\tau\partial\eta} + \frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\chi\partial\eta} - \frac{\partial f}{\partial\chi}\frac{\partial^2 f}{\partial\eta^2}\right\} \tag{19}$$

$$\frac{\partial^2\theta}{\partial\eta^2} = JaPr\left\{S\frac{\partial\theta}{\partial\tau} + \frac{\partial f}{\partial\eta}\frac{\partial\theta}{\partial\chi} - \frac{\partial f}{\partial\chi}\frac{\partial\theta}{\partial\eta}\right\}, \quad (20)$$

where the corresponding symbols are defined in the Nomenclature. The nondimensional stream function f is defined by

$$\frac{\bar{u}}{\sqrt{gLJa}} = \frac{\partial f}{\partial\eta};\ \frac{\bar{v}}{\left[\frac{Ja^3}{\gamma}\right]^{1/4}\sqrt{gL}} = -\frac{\partial f}{\partial\chi}, \quad (21)$$

where $\bar{u}$ and $\bar{v}$ correspond to the longitudinal and transversal velocity components in the physical units. The initial condition for the plate is $\theta_w(\chi, z, 0) = \theta_{w0}$. Assuming, for simplicity, the two edges of the plate to be adiabatic, the boundary conditions at both edges are given by

$$\left.\frac{\partial\theta_w}{\partial\chi}\right|_{\chi=0,1} = 0. \quad (22)$$

The other two boundary conditions for the plate are obtained from equating the heat fluxes at both lateral surfaces of the plate,

$$\left.\frac{\partial\theta_w}{\partial z}\right|_{z=0} = \frac{\varepsilon^2}{\alpha}\left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=0} \quad (23)$$

and

$$\left.\frac{\partial\theta_w}{\partial z}\right|_{z=-1} = -\frac{\varepsilon^2}{\alpha}. \quad (24)$$

The initial and boundary conditions needed to solve the foregoing equations for the condensed layer are

$$\theta(\chi,\eta,\tau=0) = \theta_{w0}, \quad f(\chi,\eta,\tau=0) = 0$$
$$\theta(\chi,\eta=0,\tau) = \theta_w(\chi,\tau);$$
$$f(\chi,\eta=0,\tau) = 0; \left.\frac{\partial f}{\partial\eta}\right|_{\eta=0} = 0 \quad (25)$$

$$\theta(\chi,\eta=\Delta,\tau) = 0; \left.\frac{\partial^2 f}{\partial\eta^2}\right|_{\eta=\Delta} = 0. \quad (26)$$

Here $\Delta(\chi,\tau)$ represents the unknown nondimensional thickness of the condensed layer to be obtained as part of the solution of the problem. The second condition of equation (26) arises from the balance of tangential shear stress at the interface [6]. The nondimensional global mass balance in the condensed layer is [15]

$$S\frac{\partial\Delta}{\partial\tau} + 3\frac{\partial\Gamma}{\partial\chi} = -\left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=\Delta}, \quad (27)$$

where Γ corresponds to the nondimensional mass flow rate of the condensed fluid and is given by

$$\Gamma(\chi,\tau) = \int_0^\Delta \frac{\partial f}{\partial\eta}\,d\eta = f(\Delta). \quad (28)$$

The system of equations (18)–(20) and (27), with the corresponding boundary and initial conditions, contains four partial differential equations with four unknowns, $f(\chi,\eta,\tau)$, $\theta(\chi,\eta,\tau)$, $\Delta(\chi,\tau)$ and $\theta_w(\chi,z,\tau)$ with six different nondimensional parameters, Ja, γ, Pr, S, α and ε. In the following sections we analyse the case for $Ja \ll 1$, with Prandtl numbers of order unity. S in general can be much lower than unity or at least of order unity. However, it will be assumed in the analysis that $S \ll Ja^{-1}$, so the quasi-steady approximation for the condensed film is justified. Large values of S are included in the analysis in order to compare with the results obtained in previous analysis [13].

For small values of Ja, the convective and the transient terms of the condensed fluid governing equations (19) and (20) can be neglected, giving the classical Nusselt's solution as

$$f(\eta) = \frac{\Delta}{2}\eta^2 - \tfrac{1}{6}\eta^3, \quad \Gamma = \tfrac{1}{3}\Delta^3$$
$$\text{and} \quad \theta_0(\chi,\eta,\tau) = (1-\eta)\theta_w(x_1 z = 0,\tau). \quad (29)$$

Therefore equation (27) transforms to

$$\frac{S}{2}\frac{\partial\Delta^2}{\partial\tau} + \frac{3}{4}\frac{\partial\Delta^4}{\partial\chi} = \theta_w(\chi, z=0,\tau). \quad (30)$$

In the following sections we analyse the cases of thermally thin and thick wall cases and explore the limits of very large and very small values of parameter α.

4. THERMALLY THIN WALL REGIME ($\alpha/\varepsilon^2 \gg 1$)

For the important case of a thermally thin wall ($\alpha/\varepsilon^2 \gg 1$), the temperature variations in the transversal direction of the plate can be neglected, as shown in Section 2. The energy balance equation (18) can be integrated in the transversal direction and after applying the boundary conditions at both faces, we obtain

$$\alpha\frac{\partial^2\theta_w}{\partial\chi^2} - \frac{\theta_w}{\Delta} = -1 + \frac{\partial\theta_w}{\partial\tau}, \quad (31)$$

with the adiabatic conditions at both edges, equation (22), and the initial condition $\theta_w(\tau=0) = \theta_{w0} = 0$. For small Jakob numbers and Prandtl numbers of order unity, the governing equations now reduce to the coupled nonlinear system given by equations (30) and (31). For the thermally thin wall regime, these equations can be integrated along the longitudinal coordinate resulting in

$$\bar{\theta}_w = \frac{S}{2}\frac{d\overline{\Delta^2}}{d\tau} + \tfrac{3}{4}\Delta_f^4 \quad (32)$$

$$\left(\frac{\overline{\theta_w}}{\Delta}\right) = 1 - \frac{d\bar{\theta}_w}{d\tau}, \quad (33)$$

where $\bar{\Phi} = \int_0^1 \Phi(\chi, \tau)\, d\chi$, for any variable and $\Delta_f = \Delta(\chi = 1, \tau)$. Δ_f is the normalized thickness of the condensed fluid at the lower edge and represents the most important dependent variable to be obtained in the present work. Δ_f is related to the condensed mass outflow as indicated by equation (29). Equation (30) also can be written as

$$\left(\frac{\overline{\theta_w}}{\Delta}\right) = S\frac{d\bar{\Delta}}{d\tau} + \Delta_f^3. \quad (34)$$

For $S = 0$, the system of equations reduce to a differential equation for Δ_f, with the solution

$$\tau = 3\int_0^{\Delta_f} \frac{s^3 ds}{1-s}$$

$$= -3\Delta_f - \frac{\pi}{2\sqrt{3}} + \sqrt{3}\tan^{-1}\left(\frac{1+2\Delta_f}{\sqrt{3}}\right)$$

$$-\ln(1-\Delta_f) + \frac{\ln(1+\Delta_f+\Delta_f^2)}{2}, \quad (35)$$

which is independent of α. Therefore, α does not have any influence on the evolution time for $S = 0$. The steady state solution, $\Delta_f = 1$, is reached for $\tau \to \infty$. In order to compare the different solutions obtained in this work, we define the evolution time as the time needed to obtain 90% of the steady final thickness of the condensed fluid at the lower edge of the plate. Thus

$$\tau_{90}(S=0) = 3\int_0^{0.9} \frac{s^3 ds}{1-s} \simeq 0.9553\ldots \quad (36)$$

and

$$\bar{\theta}_{w90}(S=0) = \tfrac{3}{4}\Delta_f^4 \simeq 0.4929\ldots \quad (37)$$

For $S = 0$, the overall plate temperature increases very slowly compared with the increase of the condensed layer thickness. Figure 2 shows the evolution

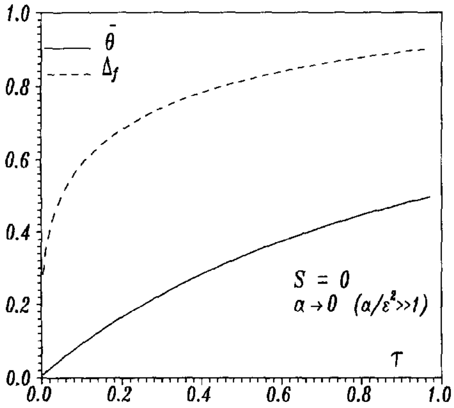

Fig. 2. Evolution of the nondimensional condensed layer thickness Δ_f and the averaged plate temperature $\bar{\theta}_w$ as a function of τ for $S = 0$, using the thermally thin wall approximation.

of Δ_f and $\bar{\theta}_w$ as a function of the nondimensional time τ obtained from equation (35).

4.1. *Asymptotic limit* $\alpha \to \infty$, $S \to 0$

This is a regular limit for α and singular for S. For very large values of the parameter α, the leading term nondimensional temperature of the plate, θ_w, changes very little (of order of α^{-1}) in the longitudinal direction, as shown in Section 2. Due to the fact that the evolution of the condensed layer mass outflow is almost insensitive to the parameter α as shown in the previous section, we retain only the leading term in α and explore the influence of the Strouhal number S on the transient process. From equation (32), we obtain after neglecting terms of order S^2

$$\Delta_f^3 = \left(\varphi - \frac{2S}{3}\frac{d\bar{\Delta}^2}{d\tau}\right)^{3/4} \simeq \varphi^{3/4} - \frac{S}{2\varphi^{1/4}}\frac{d\bar{\Delta}^2}{d\tau} + O(S^2), \quad (38)$$

where $\varphi(\tau) = 4/3\bar{\theta}_w$. From equations (33), (34) and (38), we obtain

$$\frac{3}{4}\frac{d\varphi}{d\tau} \simeq 1 - \varphi^{3/4} + S\left(\frac{1}{2\varphi^{1/4}}\frac{d\bar{\Delta}^2}{d\tau} - \frac{d\bar{\Delta}}{d\tau}\right) + O(S^2). \quad (39)$$

Assuming a solution of the form

$$\varphi(\tau) = \sum_{j=0}^{\infty} S^j \varphi_j(\tau), \Delta(\chi, \tau) = \sum_{j=0}^{\infty} S^j \Delta_j(\chi, \tau), \quad (40)$$

and introducing it into equations (39) and (30), we obtain, after collecting terms of the same power of S, the following set of equations:

$$\frac{3}{4}\frac{d\varphi_0}{d\tau} \simeq 1 - \varphi_0^{3/4} \quad (41)$$

$$\frac{3}{4}\varphi_0 = \frac{S}{2}\frac{\partial \Delta_0^2}{\partial \tau} + \frac{3}{4}\frac{\partial \Delta_0^4}{\partial \chi} \quad (42)$$

$$\frac{3}{4}\frac{d\varphi_1}{d\tau} \simeq -\frac{3}{4}\frac{\varphi_1}{\varphi_0^{1/4}} + \frac{1}{2\varphi_0^{1/4}}\frac{d\bar{\Delta}_0^2}{d\tau} - \frac{d\bar{\Delta}_0}{d\tau} \quad (43)$$

etc., with the following initial conditions

$$\varphi_i(0) = \Delta_i(\chi, 0) = \Delta_i(0, \tau) = 0. \quad (44)$$

The first term at the right hand side of equation (42) must be retained due to the singular character of the problem in this limit. The leading term solution in S has been obtained in the previous section, after solving equation (41) with the corresponding initial condition. We need to solve equation (42) in order to be able to solve the first-order equation (43). The transient term has to be retained only in a boundary layer of order S in τ. Defining the appropriate inner variables as

$$\varphi_0 = S\beta, \Delta_0^2 = 2S\phi, \tau = S\sigma \quad \text{and} \quad \chi = 3S\xi, \quad (45)$$

we obtain the inner equations as

$$\tfrac{3}{4}\beta = \frac{\partial\phi}{\partial\sigma} + \frac{\partial\phi^2}{\partial\xi}, \frac{d\beta}{d\sigma} = \tfrac{4}{3}. \quad (46)$$

The system of equations admits a similarity solution by introducing the similarity variables $\phi = \sigma^2\psi(s)$ and $s = \xi/\sigma^3$, resulting in

$$\frac{d\psi}{ds} = \frac{2\psi - 1}{s - 2\psi}, \quad (47)$$

with the condition $\psi(s \to \infty) = 1/2$. The solution is clearly $\psi = 1/2$, for $s \geqslant 1$. For $s \to 0$, the asymptotic solution is $\psi \sim \sqrt{s}$. In the outer variables, this behaviour can be written as $\Delta_0 \sim (4/3)^{1/4}\,(\tau\chi)^{1/4}$, for $\tau \to 0$, which is exactly the same as using only the outer solution. Thus, we can use the leading term solution to compute Δ_0, $\Delta_0 = \varphi_0^{1/4}\chi^{1/4}$, resulting in

$$\bar{\Delta}_0 = \tfrac{4}{5}\varphi_0^{1/4} \quad \text{and} \quad \overline{\Delta_0^2} = \tfrac{2}{3}\varphi_0^{1/2}. \quad (48)$$

Using this result, the first order equation (43), takes the form

$$\frac{3}{4}\frac{d\varphi_1}{d\tau} = -\frac{3}{4}\frac{\varphi_1}{\varphi_0^{1/4}} - \frac{2}{45}\frac{1}{\varphi_0^{3/4}}(1 - \varphi_0^{3/4}). \quad (49)$$

This equation must be solved numerically. Expanding Δ_{f0} in a Taylor series around 0.9 and retaining only the linear contribution, we can obtain finally that for $\Delta_f = 0.9$, the nondimensional time up to the first-order in S is given by

$$\tau_{90}(S) = \tau_{90}(S = 0) + Sr = \tau_{90}(S = 0) + 0.1914S, \quad (50)$$

where $r = -\Delta_{f1}/(d\Delta_{f0}/d\tau)$. Figure 3 shows $\Delta_{f0} = \varphi_0^{3/4}$, $\Delta_{f1} \simeq 3\varphi_1/(4\varphi_0^{1/4})$ and r as a function of τ.

4.2. *Solution for* $\alpha \to 0$

This is a singular limit due to the appearance of two boundary layers close to both edges in order to satisfy the adiabatic conditions. Outside of these inner zones, longitudinal heat conduction through the plate is negligible and the system of equations, up to the leading order, transform to the system of two ordinary differential equations

$$\frac{q}{p} = 1 + 3\zeta\frac{dq}{d\zeta} - q \quad (51)$$

$$q = -\tfrac{3}{2}S\zeta\frac{dp^2}{d\zeta} + Sp^2 + \frac{3}{4}\frac{dp^4}{d\zeta}, \quad (52)$$

where

$$p(\zeta) = \frac{\Delta(\chi,\tau)}{\tau}, \quad q(\zeta) = \frac{\theta_w(\chi,\tau)}{\tau} \quad \text{and} \quad \zeta = \frac{\chi}{\tau^3}. \quad (53)$$

The transformed initial conditions are $p(0) = q_0 = (0)$. For small values of ζ, the solution of equations (51) and (52) is

$$p(\zeta) = q(\zeta) \sim \zeta^{1/3} \quad \text{for} \quad \zeta \to 0, \quad (54)$$

which gives the final steady-state solution

$$\Delta(\chi, \tau \to \infty) = \theta_w(\chi, \tau \to \infty) = \chi^{1/3}. \quad (55)$$

The nondimensional thickness of the condensed layer at the lower edge ($\chi = 1$) is then

$$\Delta_f(\chi = 1, \tau) = \tau p\left(\frac{1}{\tau^3}\right). \quad (56)$$

In order to work with the relevant variables Δ_f and $\theta_{wf} = \theta_w(\chi = 1, \tau)$, we redefine the variables in the following form:

$$\Delta_f = \frac{p}{\zeta^{1/3}}, \quad \theta_{wf} = \frac{q}{\zeta^{1/3}} \quad \text{and} \quad \omega = S\zeta^{1/3}. \quad (57)$$

Therefore, the governing equations (51) and (52) transform to

$$\frac{d\theta_{wf}}{d\omega} = \frac{S(\theta_{wf} - \Delta_f)}{\Delta_f\omega^2} \quad \text{and} \quad \frac{d\Delta_f}{d\omega} = \frac{\theta_{wf} - \Delta_f^4}{\Delta_f\omega(\Delta_f^2 - \omega)}, \quad (58)$$

with the conditions $\Delta_f(0) = \theta_{wf}(0) = 1$, representing the final steady state solution. The system of equations (58) has two solutions for a given value of S. The first one corresponds to the trivial solution $\Delta_f = \theta_{wf} = 1$, which represents the problem steady solution. However, this solution cannot reproduce the initial conditions, represented by $\Delta_f \to 0$ and $\theta_{wf} \to 0$ for $\omega \to \infty$. For a finite non-zero value of S, the trivial solution is the appropriate one for $\omega \leqslant 1$. Thus, the steady state solution is achieved for a finite value of the nondimensional time τ. For $\omega > 1$, the solution bifurcates to another branch which reproduces correctly the initial conditions. This second branch corresponds to the transient solution. This can be shown using a local analysis of the trivial solution close to $\omega = 1$. Assuming

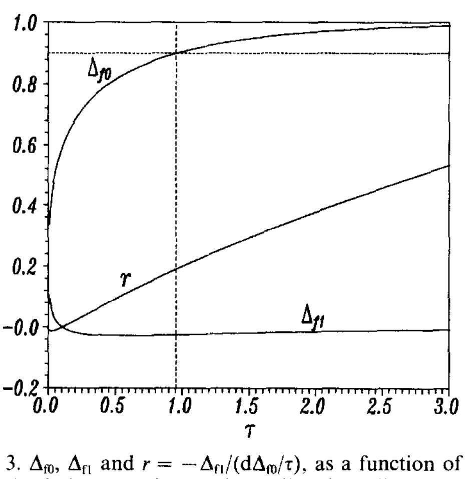

Fig. 3. Δ_{f0}, Δ_{f1} and $r = -\Delta_{f1}/(d\Delta_{f0}/\tau)$, as a function of τ in the limit $S \to 0$, for the thermally thin wall regime.

$$\theta_{wf} = 1+\varepsilon_w(\bar\omega), \Delta_f = 1+\varepsilon_\Delta(\bar\omega) \quad \text{and} \quad \omega = 1+\bar\omega, \tag{59}$$

with ε_w, ε_Δ, $\bar\omega \ll 1$, the system of equations reduces to

$$\frac{d\varepsilon_w}{d\bar\omega} \simeq S(\varepsilon_w-\varepsilon_\Delta), \frac{d\varepsilon_\Delta}{d\bar\omega} \simeq \frac{\varepsilon_w-4\varepsilon_\Delta}{2\varepsilon_\Delta-\bar\omega}. \tag{60}$$

For values of S of order unity, we obtain $d\varepsilon_w/d\bar\omega \simeq (3/4)S\varepsilon_w$. This shows that the trivial solution is the appropriate one for $\bar\omega < 0$. For large values of S, but still $S \ll Ja^{-1}$, we obtain that $\varepsilon_w \simeq \varepsilon_\Delta$ and

$$\frac{d\varepsilon_\Delta}{d\bar\omega} \simeq \frac{-3\varepsilon_\Delta}{2\varepsilon_\Delta-\bar\omega}, \quad \text{for} \quad S \gg 1. \tag{61}$$

The solution is then $\varepsilon_\Delta \simeq -\bar\omega$. For this case we can obtain a closed form solution for the transient branch as $\Delta_f = \theta_{wf} = 1/\omega$, which is clearly valid for $\omega > 1$. Thus, we recover the same result obtained in ref. [13]. Therefore

$$\Delta_f = \frac{\tau}{S} \quad \text{or} \quad \tau_{90}(S) = 0.9S. \tag{62}$$

The nondimensional averaged plate temperature is in this case

$$\bar\theta_w = \tau^4\int_0^{1/\tau^3} q(\zeta)\,d\zeta = \frac{3\tau^4}{S^4}\int_0^{S/\tau} \theta_{wf}\omega^3\,d\omega$$

$$= \frac{\tau}{S} - \frac{1}{4}\left(\frac{\tau}{S}\right)^4, \tag{63}$$

or

$$\bar\theta_{w90} = 0.735975\ldots, \tag{64}$$

which is very close to the final steady-state value of 3/4. In this case, the plate temperature evolution follows closely the evolution of the condensed layer thickness. Figure 4 shows the profiles of Δ_f and $\bar\theta_w$ as a function of τ/S for the limit $S \to \infty$.

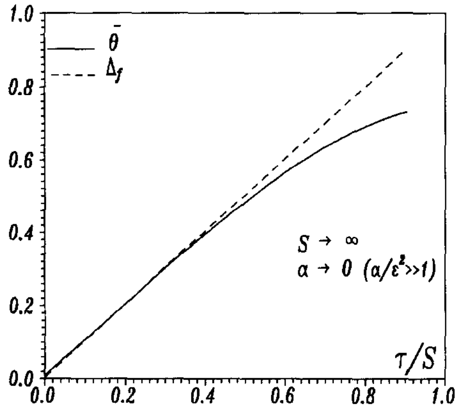

Fig. 4. Evolution of the nondimensional condensed layer thickness Δ_f and the averaged plate temperature $\bar\theta_w$ as a function of τ for $S \to \infty$, for the thermally thin wall approximation.

5. THERMALLY THICK WALL APPROXIMATION ($\alpha/\varepsilon^2 \sim 1$)

For values of α/ε^2 of order unit and $\varepsilon \to 0$, the longitudinal heat conduction is already very small and can be neglected, except in small regions close to the edges of the plate. Only the outer solution (no longitudinal heat conduction through the plate) is to be analyzed in this limit. The energy balance equation for the plate then reduces to

$$\frac{\partial^2\varphi_w}{\partial z^2} = \frac{\partial\varphi_w}{\partial\bar\tau}, \tag{65}$$

where $\bar\tau = \alpha\tau/\varepsilon^2$ is the appropriate nondimensional time for the thermally thick wall regime and $\varphi_w = \alpha\theta_w/\varepsilon^2$. Equation (65) has to be solved with the initial and boundary conditions

$$\varphi_w(\chi, z, 0) = \varphi_{w0}$$

$$\left.\frac{\partial\varphi_w}{\partial z}\right|_{z=0} = -\frac{\varphi_w(z=0)}{\bar\phi}$$

$$\left.\frac{\partial\varphi_w}{\partial z}\right|_{z=-1} = -1, \tag{66}$$

where $\bar\phi = \alpha\Delta/\varepsilon^2$. The condensed layer thickness evolution (30) transforms to

$$\varphi_w(z=0) = \frac{S}{2}\frac{\partial\bar\phi^2}{\partial\bar\tau} + \frac{3}{4}\frac{\partial\bar\phi^4}{\partial\bar\chi}, \tag{67}$$

where $\bar\chi = \alpha^3/\varepsilon^6\chi$. Therefore, the parametric set reduces in this limit to only one free parameter, the Strouhal number S. The steady-state solution is

$$\bar\phi(\bar\tau\to\infty) = \bar\chi^{1/3}, \varphi_w(\bar\tau\to\infty) = \bar\chi^{1/3}-z. \tag{68}$$

For large values of S, the system of equations is reduced to that obtained previously in Subsection 4.2. Thus, they admit the same solution given by

$$\bar\tau_{90}(S\to\infty) \sim \frac{0.9S\alpha}{\varepsilon^2} \quad \text{or} \quad \tau_{90}(S\to\infty) \sim 0.9S. \tag{69}$$

However, it should be noted that the physical evolution times for both cases are different.

6. RESULTS AND CONCLUSIONS

For values of α and S of order unity, it is necessary to solve numerically the governing equations, with the respective boundary and initial conditions. We will explore the validity of the thermally thin and thick wall approximations. The numerical scheme used here is described in the Appendix. The evolution time in physical units for the thermally thin wall regime ($\alpha/\varepsilon^2 \gg 1$) does not depend on the plate thermal conductivity and is given by

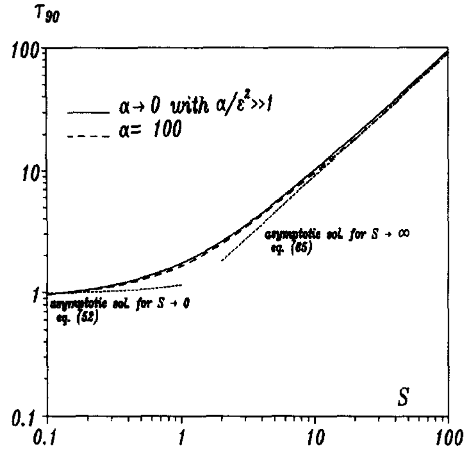

Fig. 5. Nondimensional evolution time τ_{90} as a function of S, for the thermally thin wall approximation.

$$t_{90} = \frac{\rho_w c_w h L}{\lambda_l}\left(\frac{Ja}{\gamma}\right)^{1/4} \tau_{90}(S). \qquad (70)$$

Figure 5 shows $\tau_{90}(S)$ as a function of S for two limiting values of α in the thermally thin wall regime. The two asymptotic solutions for $S \to 0$ and $S \to \infty$ are also plotted. For $S \leqslant 0.2$, the asymptotic solution $\tau_{90}(S) \sim 0.9553+0.1914S$ represents a very good approximation. On the other hand, the asymptotic solution $\tau_{90}(S) \sim 0.9S$, gives accurate results for $S \geqslant 10$. The evolution time is almost insensitive to the value of α for values of α such as $\alpha \gg \varepsilon^2$. Figure 6 shows the nondimensional evolution time τ_{90} as a function of α, for different values of the parameter S. The selected value for the plate aspect ratio ε is $\varepsilon = 0.1$. The solution for the thermally thin and thick wall approximations are also plotted. For large values of α, the thin wall approximation is appropriate, indicating that τ_{90} is almost insensitive to α. However, the thermally thick wall approximation also gives very good results, even for large values of α. The reason is because our most important parameter Δ_f is almost independent of α. Δ_f is almost independent of α, but not the Δ profiles along the plate. However, for values of S of order unity and $\alpha \sim \varepsilon^2$ ($\alpha \sim 10^{-2}$), τ_{90} begins to increase, thus indicating that the thermally thin wall approximation is not valid any more. However, the thermally thick approximation can be used for any value of α in order to calculate Δ_f. The discrepancies between the results obtained for Δ_f using both approximations (thin and thick wall regimes) disappear as the value of S increases to values much larger than unity.

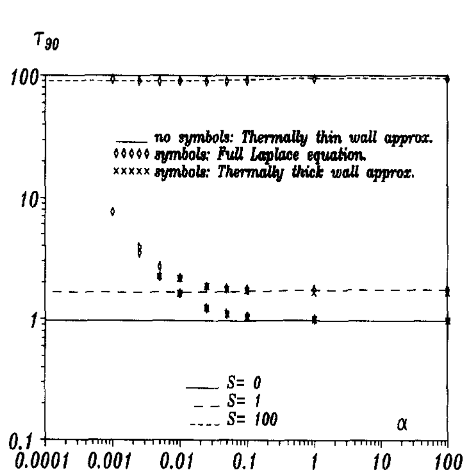

Fig. 6. Nondimensional evolution time τ_{90} as a function of α, for different values of S. The thermally thick and thin wall approximations are also shown.

In this paper, the transient condensation process of a saturated vapor in contact with one surface of a thin vertical plate has been analyzed using asymptotic, as well as numerical, techniques for small values of the Jakob number. A uniform prescribed heat flux is applied at $t = 0$, at the other vertical surface of the plate. The finite thermal conductivity of the plate material allows one to transfer heat by conduction upstream through the plate, thus changing the mathematical character of the problem from parabolic to elliptic. Assuming the plate to have adiabatic leading (upper) and trailing (lower) edges, the heat convection through the lateral surface of the plate, affected by the axial heat conduction, governs the transient evolution of the plate temperature, the condensed layer thickness and the overall condensed fluid mass outflow rate. The two asymptotic limits of thermally thin and thick wall regimes have been analyzed for this condensation process. We also explored the whole range of the Strouhal number $0 \leqslant S \ll Ja^{-1}$. We obtain in closed form the evolution time for small values of S, showing the Δ_f is insensitive to the value of α, for the thermally thin wall regime. However, there is a big influence of α, for the thermally thick wall regime. For large values of S, that is without considering the plate thermal inertia, the solution for the thermally thin and thick wall regimes give exactly the same asymptotic results. The wall thermal inertia is found to be the controlling factor for the transient condensation process in most practical cases.

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REFERENCES

1. W. Nusselt, Z., Die Oberflächenkondensation des Wasserdampfes, *Z. Ver. Dt. Ing.* **60**, 541–546, 569–575 (1916).
2. W. M. Rohsenow, Heat transfer and temperature distribution in laminar-film condensation, *Trans. Am. Soc. Mech. Engrs* **78**, 1645–1648 (1956).
3. L. A. Bromley, Effect of heat capacity of condensate, *Ind. Engng Chem.* **44**, 2966–2969 (1952).
4. E. M. Sparrow and J. L. Gregg, A boundary layer treatment of laminar-film condensation, *Trans. Am. Soc. Mech. Engrs* **81**, 13–17 (1959).
5. M. M. Chen, An analytical study of laminar film condensation, Part 1, flat plates, *J. Heat Transfer* **83**, 48–54 (1961).
6. J. C. Y. Koh, E. M. Sparrow and J. P. Harnett, The two

phase boundary layer in laminar film condensation, *Int. J. Heat Mass Transfer* **2**, 69–82 (1961).
7. H. Merte, Jr, Condensation heat transfer, *Adv. Heat Transfer* **15**, 181–272 (1973).
8. J. W. Rose, Fundamentals of condensation heat transfer: laminar film condensation, *JSME Int. J.*, Ser. 2 **31**, 357–375 (1988).
9. I. Tanasawa, Advances in condensation heat transfer, *Adv. Heat Transfer* **21**, 55–139 (1991).
10. S. V. Patankar and E. M. Sparrow, Condensation on an extended surface, *J. Heat Transfer* **101**, 434–440 (1979).
11. J. E. Wilkins, Condensation on an extended surface, *J. Heat Transfer* **102**, 186–187 (1980).
12. H. J. H. Brouwers, Film condensation on non-isothermal vertical plates, *Int. J. Heat Mass Transfer* **32**, 655–663 (1989).
13. E. M. Sparrow and R. Siegel, Transient film condensation, *J. Appl. Mech. Trans. ASME* **26**, 120–121 (1959).
14. P. M. Chung, Unsteady film condensation on a vertical plate, *J. Heat Transfer* **85**, 63–70 (1963).
15. J. G. Reed, F. M. Gerner, C. L. Tien, Transient laminar-film condensation on a vertical plate, *J. Thermophys.* **2**, 257–263 (1988).
16. F. P. Incropera and D. P. DeWitt, *Fundamentals of Heat and Mass Transfer* (3rd Edn), p. 589, Wiley, New York (1990).

APPENDIX

The heat diffusion equation (18) is numerically integrated using central differences for both spatial directions. The transient term is evaluated using the conventional Euler method. We used a quasi-linearization technique for the nonlinear terms in the evolution of the condensed layer thickness equation (30) in the form

$$\Delta^n \simeq \Delta_a^n + n\,\Delta_a^{n-1}(\Delta - \Delta_a) = n\,\Delta_a^{n-1}\,\Delta - (n-1)\,\Delta_a^n, \tag{A1}$$

where the subindex a represents the value obtained in a previous iteration. Thus the term $\partial\Delta^4/\partial\chi$ is represented as

$$\frac{\partial\Delta^4}{\partial\chi} \simeq 4\,\Delta_a^3\,\frac{\partial\Delta}{\partial\chi} + 12\,\Delta_a^2\,\Delta\,\frac{\partial\Delta_a}{\partial\chi} - 12\,\Delta_a^3\,\frac{\partial\Delta_a}{\partial\chi}, \tag{A2}$$

where the spatial derivatives are discretized using central differences, while the transient derivatives use the Euler method. A few iterations are needed to obtain a high accuracy for the results. For the numerical calculations we used a grid size of 21 points in the transversal direction z, and 71 points in the longitudinal direction, χ. The computations were done with the Cray-YMP computer at UNAM in Mexico City, using the Linpack routines.